MICHAEL DUMMETT

FREGE
Philosophy of Language

Harper & Row, Publishers
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This is the definitive work on the philosophy of Gottlob Frege, one of the fundamental figures in modern philosophy. There is no standard work on Frege, who is the source figure for Russell, Wittgenstein, and all others in modern philosophy.

Nearly ten years in the writing by Michael Dummett of Oxford, the present work deals with Frege's philosophy of language; a second volume will treat his philosophy of mathematics. The book is an exposition and an evaluation. Frege was the initiator of the modern period in the study of logic. His philosophical logic came at the time when logic was to replace epistemology (the theory of knowledge) as the starting point in philosophy. Frege's first important undertaking involved framing a formal language adequate for the expression of any mathematical statement or any train of mathematical reasoning. His explanation and justification of his views led him into profound discussions of issues of general philosophical interest, in particular the notions of the meaning of an expression and of analytic truth.

Though he is regarded as one of the great figures in philosophy of the past hundred years, Frege's life was one of disillusionment and frustration. With a few exceptions he was ignored by mathematicians and philosophers, and when he died in 1925 he was convinced his life's work had been neglected and was for the most part a failure.

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CHAPTER 14

Abstract Objects

QUESTIONS SUCH AS whether or not there are any abstract objects, what abstract objects there are, what abstract objects are and how we know that they exist, what is the criterion for their existence, where the dividing line comes between concrete and abstract objects—all these are modern questions. At first sight, such a contention appears ludicrous: one might well think such questions to be as old as philosophy. But the fact is that the notion of an 'object' itself, that is, the notion as used in philosophical contexts, is a modern notion, one first introduced by Frege. As we have seen, Frege's approach to questions of ontology involves a clean break with the tradition which had prevailed in philosophy up to his time, and which is still exemplified by such works as Strawson's *Individuals*. According to the ancient tradition, entities are to be categorized as particulars and universals. It is characteristic of particulars that we can only refer to them and predicate other things (universals) of them—say things about them: we cannot predicate them of anything else—we cannot, as it were, say them of anything. Universals, by contrast, can both be predicated of particulars, and also referred to in the course of predicating other things (higher universals) of them. Thus, on this tradition, a universal can be alluded to ('introduced', in Strawson's terminology) in two different ways: both by a predicative expression, by means of which we predicate the universal of something else; and by a term, by means of which we refer to the universal in the course of predicating something of it.

Of course, we can, in full accordance with this tradition, distinguish between those terms which we use to refer to particulars and those which we use to refer to universals. But we shall be unable to discern what differentiates universals from particulars if we concentrate only on such terms; for the difference lies precisely in the fact that we can predicate universals of other things. Hence, if we want to understand what universals are, we have to concentrate on the predicative expressions by means of which we predicate them of other things: it is by a study of the character of predication that we shall come to understand the essential nature of universals.

For Frege, as we have seen, this approach is fundamentally misconceived. Terms (proper names) and predicates are expressions of such radically different kinds, that is, play such radically different roles in the language, that it is senseless to suppose that the same thing could be alluded to both by some predicate and by some term. It is true enough that we can grasp the sort of thing which a predicate stands for—a concept—only by understanding the linguistic role of a predicate: but just for this very reason we can never conceive of an expression as standing for a thing of that kind if the expression was incapable of playing that linguistic role.

This contention is not, of course, a mere fiat on Frege's part: he is not arbitrarily choosing to lay it down that the referent of a term shall not be recognized as coinciding with the referent of a predicate. Still less is it a question of mere lack of mental agility—that Frege is incapable of grasping a conception to which philosophers with a more flexible imagination can attain. Rather, Frege is denying that it is possible, on the traditional basis, to construct a workable semantics for a language: we can do nothing with the suggestion that a certain term—say, 'wisdom'—should be regarded as standing for the very same thing as that which a certain predicate—in this case, 'ξ is wise'—stands for. In order to make use of that suggestion, we should have to be able to construct from it an account of the truth-conditions of sentences in which the abstract term occurred, e.g. of a sentence like 'Wisdom is not confined to the old' or 'Wisdom depends on experience'. That would mean explaining the predicate in such a sentence ('ξ is not confined to the old' or 'ξ depends on experience') as applying to something itself given as the referent of a predicate. But that would merely mean construing the sentence in which the abstract term occurred as the equivalent of one containing the corresponding predicate: e.g., construing 'Wisdom is not confined to the old' as an idiomatic variant of 'Not only the old are wise'. There is, of course, no absurdity in such a reconstrual: as we have seen, an acceptance of Frege's notion of objects does not require us uncritically to admit every abstract noun as a genuine singular term; more probably, we shall want to regard most of them as merely used to form idiomatic variations on sentences containing the corresponding predicate or relational expression. But to give such an account of an abstract noun is precisely to deny it the status of a genuine term or proper name: and just for that reason there is no such thing as allowing abstract nouns as real singular terms, but at the same time assigning to them the same reference as that possessed by the corresponding predicate. Seen in this light, the traditional conception is simply incoherent.*

It would be possible to accept Frege's notion of an object, characterized as the sort of thing which can be the referent of a proper name, where it is assumed that the referents of proper names and of incomplete expressions are of radically distinct kinds, and yet to disbelieve in the existence of abstract objects. Such a position is, in fact, nominalism in the latter-day sense, that is, in the sense used by Goodman and Quine. (Nominalism in its original sense meant the denial of the existence of universals, that is, the denial of reference either to predicates or to abstract nouns: in Goodman's sense it means the denial of the existence of abstract objects.) But, because the notion of 'objects', in general, only has its home against the background of Frege's radical distinction between objects and concepts, even the question whether there are abstract objects requires this background for its formulation.

The fundamental question of ontology is, 'What is there?', where, of course, since an actual inventory is not required, the intention of the question is, 'What kinds of thing are there?'. How the question is broken down then depends upon the basic principles of categorization. On the traditional conception, the first step towards breaking it down consists of specializing to the two questions, 'What particulars are there?' and 'Are there universals, and, if so, what universals are there?', where, of course, the second question raises the problem of nominalism as traditionally conceived. The question, 'What objects are there?', on the other hand, arises only against the background of a Fregean ontological perspective, and its companions are, 'Are there concepts?', 'Are there relations?', 'Are there functions?' and 'Are there truth-values?'. The question about objects then may be broken down further into ones concerning concrete objects and abstract objects.

The argument given above against the traditional conception under

* It might be thought that a further objection could be grounded on the different truth-conditions for existential quantification over concepts and over abstract objects. Frege insisted that a self-contradictory predicate was nevertheless meaningful, and may be used to construct true sentences: so from 'No man is both wise and foolish' we may validly infer 'There is something which no man is'. If, on the other hand, we believe in qualities, conceived of as universals, i.e. as the referents of abstract nouns construed as singular terms, we shall be likely to require a quality to be logically capable of exemplification, and thus to say, e.g., that there is no such quality as simultaneous wisdom and foolishness. This is not, however, an impressive argument. On any view, we want to allow a sense under which the remark, 'There is no such thing as being both wise and foolish', is true; and, if this is understood in terms of second-order quantification, then that quantification must be taken as restricted to concepts which have application, or to concepts which logically could have an application.

which a predicate and the corresponding abstract noun are taken as having the same reference depends upon the acceptance of the outlines of a Fregean semantics. With anyone who assumes that the correct semantics for natural language is of some entirely different kind, the argument will carry no weight. Given that Frege's analysis of language, and the semantics that goes with it, are basically correct, however, the notion of an object, as introduced by Frege, is evidently the fundamental one required for the study of ontological questions. The notion of an object thus stands or falls with the supposition that Frege's analysis of language—that, namely, which is enshrined in standard classical predicate logic—provides the foundations for the semantics of natural language. There are, of course, many features of natural language with which Frege did not deal: but Frege's faith was that these features could all be fitted in to the general framework that predicate logic supplies, and this is also the faith of all philosophers of language, such as Quine and Davidson, working within the Fregean tradition; certainly we have no other general framework available. The notion of an object has been given sense only against the background of an analysis of language of Frege's kind, and the question, 'What objects are there?', like the more specialized question, 'Are there abstract objects?', arises only against this background.

The notion of an object plays within Frege's semantics a twofold role. On the one hand, objects are the referents of proper names: the truth-conditions of sentences containing proper names, in particular, of atomic sentences, are to be explained in terms of the relation of reference between proper names and the objects for which they stand. Equally, of course, objects are what predicates are true or false of. While we are concerned solely with atomic sentences and combinations of these by means of the sentential operators, we need have the conception of a predicate's being true or false of an object, or a relational expression's holding or failing to hold between a pair of objects, only for simple predicates and relational expressions. It must be extended to complex predicates when we come to the second of the roles played by the notion of an object, namely the account of quantification: objects are required to compose the domains of quantifiers, that is, the ranges of the individual variables which can be bound by quantifiers. However much it may be necessary to depart from Frege's ideas in detail, if Frege's analysis of language is correct in principle, the notion of first-order quantification will be an indispensable tool for the analysis of many sentences of the language, and, wherever first-order quantification is involved, it must be possible to specify a suitable totality of objects as the domain of quantification.

When Frege is arguing that things of a certain kind, for instance numbers, must be taken as objects, he tends to concentrate on the form of expressions standing for those things, that is, to concentrate on the first of the two roles of the notion of an object. Thus, at a crucial point in *Grundlagen der Arithmetik*, Frege wishes to establish that (cardinal) numbers are objects: and he does so by concentrating attention on number-words used as nouns and on numerals as occurring in most arithmetical contexts, and urging that these have to be construed as proper names, i.e. singular terms. It is then fairly easy for him to make a highly plausible case for this contention, e.g. that '5' in '5 is a prime number' or in '$5 \times 2 = 10$' functions as a term, or that 'ten million' does the same in 'The population of Tokyo is ten million'. But the importance of the question hinges much less on the construal of '5' or of 'ten million' in such contexts as proper names, than on the fact that, once he has to his satisfaction established that numbers are objects, he then takes it for granted that they may be regarded as falling within the range of his individual variables; and, in particular, that, where numerical terms are taken as formed ultimately by means of the term-forming operator 'the number of x's such that $\Phi(x)$', we may legitimately fill the argument-place of this operator with a predicate defined over numbers.

Frege made the natural assumption that it is possible to take one single maximal domain, the domain of all objects, as being, in all contexts, the domain of the individual variables. This was a natural conclusion for him to draw from the simple observation that the effect of a restriction of a domain may always be achieved by the use of a predicate satisfied by all and only the members of that domain: 'All men are brave' may be rendered by '$\forall x$ x is brave', where the range of the variable is taken as being the set of men, but equally by '$\forall x$ (x is a man $\rightarrow$ x is brave)', where 'x' is taken as ranging over any set which includes the set of men; equally, 'Some men are honest' may be rendered either by '$\exists x$ x is honest', where 'x' ranges over the set of men, or by '$\exists x$ (x is a man & x is honest)', where 'x' ranges over a more inclusive set. What more obvious than to suppose that individual variables may always be taken uniformly to range over a single most inclusive totality, intended restrictions being effected always by suitable predicates? That is why Frege proceeds without further argument from the thesis that numbers are objects to the consequence that any individual variable may be taken as ranging over, among other things, numbers. Hence for Frege an interpretation of a formalized language requires only a specification of the non-logical constants: there is no need, for him, to specify especially the domain of the individual variables, since this has been taken once for all as the totality of all objects. This of course contrasts with the standard modern notion of an interpretation, which does demand that we first fix a domain.

We neither need nor can follow Frege in supposing that one single all-embracing domain will serve for all uses of individual variables: for the most direct lesson of the set-theoretic paradoxes is that, at least when we are concerned with abstract objects, there is no one domain which includes as a subset every domain over which we can legitimately quantify: we cannot give a coherent interpretation of a language, such that every sentence of the language can be taken as having a determinate truth-value, by taking the individual variables to range over everything that answers to the intuitive notion of a set, or that of a cardinal number or that of an ordinal. We must therefore separate Frege's basic intuition, the use of quantification understood as relative to a determinate domain as a fundamental tool in the analysis of language, from his incorrect further assumption, that this domain, by being stipulated to be all-inclusive, can be taken to be the same in all contexts.

Quine is celebrated for a thesis, expressed by the slogan 'To be is to be the value of a variable', about what we are to regard as being the ontological commitment of a language. The thesis amounts to this: that, in order to determine what objects the use of some segment of our language commits us to the existence of, we have to enquire how to analyse that language in terms of predicate logic (allowing the possibility that the analysis will require the use of a many-sorted theory, employing several distinct styles of individual variables): the objects to which we are committed will then be those comprising the domains of the different sorts of individual variable under our analysis.

In some of the earlier formulations Quine sometimes writes as if ontological commitment were carried only by existential quantification, even, perhaps, initial existential quantification, i.e. as if we could settle the question to the existence of what objects we are committed by considering which statements we are prepared to assert of the form 'There exists . . .'. Clearly, however, it is evident that we cannot restrict attention merely to initial existential quantification: a statement of the form '$\forall x \exists y\ B(x,y)$' has existential import just as much as does one of the form '$\exists x\ A(x)$'. But, in any case, existential quantification is no more to the point than universal quantification. If we are giving a semantics for our language by analysing it in terms of the language of predicate logic, and then providing a semantics of the classical kind for the analysed language, we need to specify domains for each sort of individual variable in order to determine truth-conditions for sentences involving, under the analysis, first-order quantification of whatever kind: hence, if the analysis is correct, our ontological commitment comprises all the objects of any of these domains.

Quine sometimes states his thesis as if we were already provided with a number of theories formulated within the framework of predicate logic, i.e. as if the question of ontological commitment could arise only once some analysis of informal language was already given. At first sight, this makes ontological questions seem quite unproblematic: if we are considering only formalized languages, then what objects could a theory formulated in such a language commit us to save those within the domain or domains of its individual variables? Quine of course recognizes that ontological questions may be problematic, and views this as a matter of when a theory formulated in one language may be replaced by another theory formulated in another language: the problem now is to explicate the relevant notion of 'replacement'. This is hard to do without reference back to the informal language which we are trying to analyse: the sense in which one theory may be said to replace another that is relevant to ontological questions is that in which it replaces it as a proposed analysis of a segment of informal language. At least, this is so when the question is whether or not certain entities, e.g. propositions, can be 'eliminated'; what is really at issue here is whether or not there are sentences of our language a correct analysis of which demands that we construe certain expressions occurring in them as standing for propositions. (The matter stands rather differently when the question is one of the 'reduction' of one class of objects to another class, e.g., to give the classic instance, of the class of ordered pairs $\langle x,y \rangle$ to the class of sets of the form $\{\{x\}, \{x,y\}\}$. Here there is no question of a more correct or more workable analysis, but merely of an equally serviceable one which achieves an economy by mapping one domain into another.)

Ontological questions can thus certainly arise in the absence of an accepted analysis of a given fragment of informal language; and, indeed, they may constitute precisely a way of asking what form the desired analysis should take. What is the case, however, is that a question as to what objects some fragment of language commits us to take as existing presupposes a prior assumption that the general framework within which the correct analysis is to be given is that of the language of predicate logic: for it is only in relation to an analysis of that form that we understand the notion of an object. Quine's thesis has been the object of a number of criticisms and attacks, most of which simply miss this point: it is only in the context of a Fregean type of analysis of language that the question what objects we commit ourselves to can arise, and, within this context, there is no room for argument about whether the crucial question is as to the domains of the individual variables, because it is by reference to the Fregean semantics for first-order quantification that the notion of 'object' which we are using has been given in the first place.

That ontological questions can be problematic follows, of course, from the fact that Frege's analysis of language purports only to reveal its so-called 'deep' structure, not its surface structure. Palpably, the sentences of a natural language are not constructed in such a way as to reveal on their face that they exemplify the pattern of sentence-formation enshrined in predicate logic. Frege's philosophy of language embodies the faith that it is only by representing the various linguistic devices belonging to the natural language as constituting often far from transparent means of expressing sentence-forming operations of predicate logic that we shall attain an adequate semantics for our language. This faith rests on the partial success of Frege and of others working in that tradition in accomplishing such a programme for analysing natural language, and partly on the absence of any alternative general model of analysis. There remain many problem areas: ontological problems about whether there are propositions, events, etc., reflect remaining uncertainties about how to carry out the programme in such areas.

Quine concentrates upon the second role of the Fregean notion of 'object', namely in explaining first-order quantification, because he does not take seriously its first role, in explaining the use of proper names. His reason for not taking this seriously is that he thinks we can get on without proper names altogether, construing all apparent proper names as definite descriptions, and then analysing the latter in terms of quantification in the Russell manner. Such a proposal does not in fact effect any significant economy, because, even if a language were viewed as not really containing any atomic sentences (because it contained no genuine singular terms), it would be necessary, in order to give the semantics of such a language, to specify, for each predicate, when it was true of any given object (such a specification doing the work of a stipulation of truth-conditions for the atomic sentences). Quine has even coupled this thesis with the assertion that, where we have no infinite domains, quantification can be eliminated in favour of finite disjunction and conjunction. It is, of course, true that, if the individual variables of a language range over a finite domain, and the language contains terms for every element of the domain, then every sentence of the language is equivalent to a quantifier-free sentence. But if, after having in this manner eliminated the quantifiers in favour of finite combinations of atomic sentences, we then eliminate the atomic sentences by construing the singular terms occurring in them as Russellian definite descriptions, we have reintroduced quantification: the process thus seems merely circular, and it is hard to see how Quine can argue on this basis that such a language is free of ontological commitment altogether.

Quine's assumption that the question, 'What objects are there?', exhausts the content of the general ontological query, 'What is there?', is, on the other hand, in sharp contrast with Frege's view. Such a reduction depends upon holding, as Quine does, that all higher-order quantification can be dispensed with in favour of quantification over abstract objects, namely classes. From Frege's standpoint, such a claim would be absurd, because for him the notion of a class was itself a second-order notion: that is, while for Frege classes are indeed objects, so that quantification over classes requires only first-order quantification, we cannot explain what a class is, or, specifically, define the fundamental relation of membership in a class, without quantification over concepts. Frege's formalization of the theory of classes takes as primitive the class-abstraction operator, 'the class of x's such that $\Phi(x)$', by means of which terms for classes are formed, and defines the membership relation in terms of it; whereas, of course, modern axiomatized set theory takes the membership relation as primitive, and defines the class-abstraction operator in terms of that (by means of a description operator or of a convention for the definition of terms). The possibility of employing a formalized first-order theory of this kind does not, indeed, end the argument: it is still necessary to enquire to what extent a first-order theory succeeds in capturing the intuitive notion of 'class'. Quine emphatically holds that it does, while Frege, presumably, would deny this: but we shall not here further pursue this question, which belongs more to the philosophy of mathematics. At least this much is clear: given the correctness in principle of a Fregean analysis of language, our ontological commitment depends principally upon the types of quantification which are to be employed in our languages as so analysed. Quine would make the correlation a very direct one, namely that, if only various sorts of first-order quantification are needed, then we shall be committed only to the existence of the corresponding sorts of objects; but that, if higher-order quantification is needed, then we shall also be committed to the existence of suitable ranges of concepts, relations and functions. On Frege's view, this assessment of ontological commitment is excessively parsimonious, and it will be argued in the next chapter that, in this, Frege is right as against Quine. From Frege's standpoint, the matter should be expressed thus: that our ontological commitment depends upon what expressions of our language (including incomplete expressions) have to be taken as forming logically significant units and therefore as having reference; and this will in turn depend upon the analysis we give of the sentences of the language, and, specifically, upon what kinds of quantifiers and other second- or higher-level operators such analysis involves. The general point, however, is common to Frege and to Quine: the ontological commitment embodied in a language depends upon its quantificational structure, as revealed by logical analysis.

By implication, Frege recognizes the possibility of drawing a distinction between concrete and abstract objects: that is to say, he employs, in *Grundlagen*, the notion of 'concrete' (*wirklich*, literally 'actual') objects, though only in the course of arguing that not every object is concrete. He puts the distinction to no work, however: for him, abstract objects are just as much objects as concrete ones, and may just as legitimately be taken as the referents of proper names or as belonging to the domain of first-order quantification, and that is an end of the matter. In post-Fregean philosophy, however, abstract objects have been very much a bone of contention. As already noted, some philosophers, notably Nelson Goodman, have propounded a new variety of nominalism, according to which reference to or quantification over abstract objects is not properly intelligible, and must always be eliminated or replaced by some other form of locution. Others have felt unable to espouse such total puritanism, but have held, nevertheless, that the 'countenancing' of abstract objects is a grave step, intellectually sinful if undertaken without necessity, and have therefore seen reductionist devices which allow the elimination of reference to or quantification over them as something which it is a major aim of philosophy to construct. Some, again, have felt that the recognition of abstract objects is tolerable only when they are construed as 'posits', or have argued that it is permissible only because concrete objects too are really posits. Controversy in the philosophy of mathematics has frequently turned on whether mathematical objects, such as natural numbers, real numbers and sets, are to be regarded as independently existing abstract objects or as free creations of the human mind, or, again, as dispensable altogether.

If we are to make any progress in assessing Frege's ready admission of abstract objects as harmless, a regrettable necessity or the primal philosophical sin, we must have some means of supplying some rationale for the distinction between concrete objects and abstract ones. The rough, everyday, distinction between them draws the line according as they are or are not accessible to the senses: 'abstract nouns in *-io* call feminina one and all; masculine will only be things that you can touch and see', as the gender rhyme has it. This makes the distinction relative to human sensory faculties, for it is evidently sometimes a contingent matter whether or not something affects human sense-organs: by such a criterion, light-waves would be concrete but radio waves abstract. Moreover, difficulties would arise, in applying such a definition, as to what counted as, e.g., 'feeling' something: do we feel the gravitational pull of the Earth, for instance, or do we feel only the pressure of the objects which support us, or, again, do we really feel, not the pressure, but only the objects themselves?

The distinction is evidently connected with that which we noted previously, between those objects which can, and those which cannot, be the objects of an ostension: those which can be referred to by means of a demonstrative accompanied by a pointing gesture (as opposed to the use of a grammatical demonstrative merely to pick up a reference from a previous sentence), and those which cannot be referred to by such means. When we use a demonstrative to refer ostensively to an object,* the context must, indeed, supply an appropriate criterion of identity, or else one must be expressly given by means of a general noun: lacking a criterion of identity, we should not know, as it were, along what plane to slice in order to detach the object from its environment. But we have seen that there are objects which do not permit themselves to be pointed to at all, even when an appropriate criterion of identity is provided. Shapes and directions were prime examples of objects of this kind: a shape has to be given as the shape of some two- or three-dimensional region, a direction as the direction of a line or movement. In such cases, the mere use of a demonstrative, accompanied by a pointing gesture, will not be enough to determine the object referred to, even when the associated criterion of identity is supplied: we have, e.g., to know not only that it is a shape that is being referred to, but how to identify that of which it is being specified as being the shape. Thus an expression which stands for a shape will characteristically employ the functional expression 'the shape of ξ': even when the term is not itself complex, it will have been introduced as the equivalent of some term formed by means of that functional expression.

We evidently have here an adumbration of the distinction between concrete and abstract objects; but it needs considerable refinement before it can be regarded as adequate. The class of concrete objects cannot be simply identified with the class of objects which can be the object of an ostension, at least if ostension is taken literally as involving the use of a pointing gesture: such a characterization would include visual objects, such as rainbows, and opaque material objects, but not, for instance, a colourless gas, a sound or a smell. Furthermore, we have no assurance that the positive characterization we have given of abstract objects—those falling within the range of some function like 'the shape of ξ'—covers the entire complement of the class

* We shall note in Chapter 16 that by no means every ostensive use of a demonstrative is to refer to an object.

of objects capable of being pointed to, even under a suitably extended sense of 'pointing'.

Let us first make more precise the conception of a functional expression which is like 'the shape of ξ' in the relevant ways. It makes no difference whether the functional expression is of first or second level: for our purposes the expression 'the number of Φ's' ('the number of x's such that $\Phi(x)$') belongs to the type we are interested in. On the other hand, 'the capital of ξ' emphatically does not, since the introduction of this expression into the language does not serve to introduce a new range of objects: that is, it is part of the intuitive meaning of 'the capital of ξ' that an object within its range is one that can be referred to (e.g. as a city with a certain geographical location) without any use, direct or indirect, of the notion of the capital of a country. We laid it down as a necessary condition for a functional expression to be of the type in which we are interested that its intuitive meaning should not require us to identify objects denoted by a completion of that functional expression with objects referred to by a term not involving that expression. For instance, while it is possible to identify cardinal numbers with certain classes, and thus, relative to such an identification, to refer to some cardinal number by means of a term which in no way involves the functional expression 'the number of Φ's', such an identification is not demanded by the intuitive meaning of the expression 'the number of . . . '. But suppose now that someone claims that names of numbers need not be thought of as involving the function 'the number of Φ's', even when no such extraneous identification is made. We cannot avoid allowing that the name '$\aleph_0$' has as its intuitive meaning 'the number of natural numbers'; and, given the close connection in our language between the finite numbers and the procedure of counting, it is highly plausible that the fundamental intuitive meaning of, say, '67', taken as denoting a cardinal number, is 'the number of numerals from "1" to "67" '. But we can easily conceive of a linguistic community who have no definite number-words other than '0', '1', '2', '3' and '4'. Such a supposition does not deny to the members of this community a grasp of the basic notion of cardinality, that of a bijective mapping or one-one correspondence. They may perfectly well know how to establish, for finite sets or even for infinite sets as well, that one set has fewer members than, or just as many members as, another set, by appeal to the notion of one-one correspondence. They might, by the use of tallies, keep a record of how many things of a certain kind there were, so that, e.g., a shepherd could check whether all of his sheep were in the fold by pairing them against the notches in the tally-stick he kept for his flock. (There is a frequently repeated story—I have no idea whether it is true or not—that various primitive peoples have no number-words for numbers greater than four: but it is always assumed by those who reiterate this story that such peoples must be virtually innocent of the notion of cardinality. As we see from the case described, there is no warrant at all for such an assumption.) The only difference between such people and ourselves is that they have not hit upon the idea of having an arbitrary infinite sequence of expressions to use as a common standard or tally, and thus supply definite answers in all finite cases to the question 'How many?'.

The point of imagining such a community is to supply a language within which there are names of cardinal numbers, namely the words for the numbers from 0 to 4, but it may be claimed that these names are not explained in their language, even tacitly, as the equivalents of expressions formed by means of the operator 'the number of Φ's'. They do not count, but can recognize straight off, as we can do without counting, how many things of a given kind there are when there are less than five of them: so there is no prototypical three-membered set such that the word for 'three' in that language may be said to mean 'the number of members of that set'.

Clearly there is no case for saying that, in such a situation, the cardinal numbers would be being thought of as concrete rather than abstract objects. The interest of this objection lies only in its forcing us to make more precise the stipulation that the intuitive meaning of the functional expression should not require the identification of objects referred to by means of it with objects referred to in some other way. There is a close parallel between the names of numbers, as used in a linguistic community of the kind imagined, and our use of colour-words as nouns, i.e. as the names of colours. Blue is the colour of the sky, but it cannot be maintained that the name 'blue', as ordinarily understood, involves implicitly a reference to the sky or to any other visual or material object as being that of which the object it stands for is the colour. But this fact is not, in itself, a reason for denying the status of abstract objects to colours.

Now if we are to take the example seriously, we must suppose that, in the community imagined, there is a serious practice of using certain singular terms as names of cardinal numbers: their language must contain an operator with the meaning 'the number of Φ's', and they must have some form of arithmetic, that is, a vocabulary for properties of, relations between and functions of cardinal numbers (perhaps only the finite ones); if they were to use number-words only in the contexts in which we use number-words as grammatical adjectives, that is, as answers to questions of the form 'How many Φ's are there?', then they could not be said to refer to numbers as objects at all, and the question whether they took them as abstract objects or concrete would not arise. Thus, even if they have no conception of infinite cardinals, they are somewhat in the position which obtains for infinite cardinals in the absence of the Axiom of Choice: we may suppose that with each set is associated an object as its cardinal number, but we have no systematic way, without appeal to the Axiom of Choice, of selecting a representative set of each cardinality; likewise these people might use variables ranging over cardinal numbers, or over finite cardinals, and formulate, or prove, laws such as the commutative law for addition, etc. It is unthinkable that, where the words for numbers up to four were the only determinate number-words in the language, there should be a serious use of these words as names of numbers (as opposed to a use corresponding to our employment of number-adjectives) without a background such as this.

Now the reason for saying that numbers are abstract objects rests on the character of the transition from giving answers to the question 'How many?' to speaking of numbers as objects. It is quite irrelevant that, for speakers of this language, a statement of the form 'There are three *F*'s' is not, even implicitly, based upon a correlation between the *F*'s and some distinguished three-membered set. What matters is that '3', when used as the name of a number, is understood as standing for something within the range of the numerical variables, i.e. for something which is, for some concept *F*, the number of *F*'s (of objects falling under *F*). We might express this by saying that '3', understood as a singular term, is for these people explained as meaning 'the *n* such that, for any *F* such that there are just three *F*'s, *n* is the number of *F*'s'. It is quite unimportant what procedure they use for determining, for a given concept *F*, that there are, or are not, just three *F*'s—that they tell this by immediate inspection rather than by counting. This has as a consequence only that '3' is not explained for them as meaning 'the number of *G*'s' for any specific concept *G*: but their use of '3' as the name of a number depends upon their recognizing it as standing for something within the range of the functional expression 'the number of Φ's'. We could not say that they did use '3' as the name of a number unless they took it as a possible value of a numerical variable; and the only possible explanation of the intended range of the numerical variables is precisely as the range of that functional expression. Hence it remains the case that, for these people as much as for us, the use of a term as the name of a number can be explained only in terms of the reference of terms formed from the functional expression 'the number of Φ's', even though the meaning of the numerical term is not explained in the simple manner as equivalent to a specific completion of that functional expression.

In exactly the same way, as remarked above, a similar objection would not suffice to show that colours are not abstract objects. The word 'blue', when used as a noun, i.e., as the name of a colour, rather than as an adjective, i.e., as a predicate applying to material or visual objects, does not mean anything as simple as 'the colour of the sky': but, in just the same manner, we might explain it as meaning 'the c such that, for every x, if x is blue, then c is the colour of x'. What is significant is, once again, not that we are capable of assigning material and visual objects to equivalence classes under the relation of matching in colour otherwise than by referring to individual representative members of the equivalence classes, and so of specifying the colour common to the members of such an equivalence class without designating it as the colour of any particular object, but the transition from using colour-predicates and relations of colour between material objects to using colour-names by which to refer to colours as objects.

It is essential to the understanding of '3', used as a singular term, that we recognize that there are kinds of thing such that '3' stands for the number of things of any one of those kinds; and, as we acquire the use of a colour-word like 'blue', used as a noun, that we understand that there are material or visual objects such that 'blue' stands for the colour of those objects. By contrast, it is quite inessential to a grasp of the use of a name like 'Madrid' that we are aware that there is a country of which 'Madrid' stands for the capital. So we may modify our requirement that, for an object to be abstract, there must be a functional expression such that that object cannot be referred to save as the referent of a term formed from that functional expression; instead, we require only that an understanding of any name of that object involves a recognition that the object is in the range of that functional expression.

Nevertheless, as we saw earlier, there is good ground for denying the status of abstract objects to colours. In our language, as it is actually learned, reference to colours as objects comes in at a fairly late stage. A child first learns to use demonstratives, proper names and other terms to refer to material objects and to visual objects like the sky, and to use various predicates and relational expressions applying to such objects, including colour-adjectives and expressions like 'ξ is the same colour as ζ' and 'ξ is darker than ζ'. Only much later does he acquire the use of colour-words as nouns, i.e. as proper names of colours, and of the functional expression 'the colour of ξ', together with predicates and relational expressions applying to colours, like 'ξ is primary' and 'ξ is complementary to ζ'. But this order of acquisition of concepts, though no doubt a psychological necessity, is not epistemologically necessary. There is no logical absurdity in the supposition that colour-words, used as nouns, should be introduced into the language before there was any means of reference to material or visual objects, and that these names of colours were introduced by ostension: the child would learn the procedure of pointing and asking, 'What is that?', and would associate with the answer (e.g. 'Crimson') the criterion of identity for colours. As we have noted, the construal of the colour-word given in answer as a proper name rather than an adjective depends upon the child's acquiring simultaneously a vocabulary of predicates and relational expressions applied to colours; but there is, again, no logical absurdity in supposing him to do so before he learns to make reference to material and visual objects. The reason that this can be conceived for colours, but not for shapes, is, as we have seen, that, in order to establish what colour is being referred to, we need only to determine the direction of the pointing finger, whereas, in order to determine the shape referred to, we need some means of circumscribing the region of which it is being taken to be the shape.

Our criterion for objects of a certain kind being abstract rather than concrete was that there should be some functional expression such that it was essential, for the understanding of any name of an object of that kind, that the referent of the name be recognized as lying within the range of that functional expression. (Under this formulation, it is no longer necessary to make any restriction on the kind of functional expression in question, since the definition itself rules out those like 'the capital of ξ'.) From the case of colours, we see that we should not construe the necessity invoked in the definition as relative to a particular language or a particular order of acquisition of the different parts of the language: it must govern the use of any terms, in any language, for objects of the given kind. It may reasonably be said of an English speaker that making the transition from the use of colour-words only as adjectives to their use as nouns as well involves an acquisition of the use of the functional expression 'the colour of ξ', and that it is therefore essential, for such a speaker to understand the meaning of a colour-word used as a noun, that he takes it as standing for something in the range of that functional expression. But this is not enough to lead us to classify colours as abstract objects, since we can conceive of a language containing colour-words used only as nouns, and containing no expression playing the role of 'the colour of ξ', since containing no terms which could fill its argument-place.

Colours are, indeed, on the borderline between concrete and abstract objects: they would certainly be classified as universals on the traditional view, and a slightly different way of construing the criterion we have been using would put them on the abstract side of the fence. It remains that there is a highly important distinction between colours on the one hand and such things as shapes and directions on the other, a distinction that we may express by saying that a shape has to be taken as the shape of something and a direction as the direction of something, but a colour need not necessarily be understood as the colour of anything: our result, that colours are to be regarded as concrete objects but shapes as abstract ones, agrees with those of Goodman and Quine, and for much the same reason. The sense in which a shape or a direction must be 'of' something is very akin to the conception of logical dependence which Aristotle expresses by the preposition 'in' when he gives as part of his characterization of a substance that it is not 'in' anything else. Of course, it may be objected that there are shapes such that there is nothing of which they are the shape, so that we ought not to claim more than that, for any shape, there *could* be something of which it was the shape: but here we encounter the question as to the criterion of existence for abstract objects, which we shall treat of later, when we have settled the present question, how to classify objects as concrete or abstract.

The condition we have formulated for being an abstract object is certainly sufficient, but it does not appear to be necessary. Frege cites, as a typical example of an abstract object, the centre of mass of the Solar System. The functional expression, 'the centre of mass of ξ', is certainly not one of the kind we have been considering, but resembles 'the capital of ξ': a centre of mass is a point, and there are many ways in which that point might be referred to otherwise than as the centre of mass of anything. Points, on the other hand, seem eminently to be candidates for the status of abstract object: but it is difficult to find a functional expression which is related to points as 'the shape of ξ' is related to shapes.

A moment's thought will yield a multitude of further examples: things to which we give names, names which certainly function in sentences as singular terms, things which have reasonably well-defined criteria of identity and which can be counted, but which do not appear to lie within the range of any functional expression of the kind Frege considers when discussing the construal of numbers as objects. Conventions in Bridge and openings in chess have what are indisputably proper names in the strict sense, such as 'Solid Suit Convention', 'Blackwood', 'Sicilian Defence', 'Giuoco Piano'. Games themselves have proper names, but are hardly concrete objects: an evening's play at Poker might be classified as an event, and thus a concrete object, but the game of Poker itself is as much an abstract object as the letter A. Indeed, as we reflect on the variety of objects that can be named, the dichotomy between concrete and abstract objects comes to seem far too crude: to which of the two categories should we assign the Mistral, for instance? This feeling, of an unsurveyable multiplicity of types of object, naturally reinforces Frege's contention that the distinction between concrete and abstract objects is not of fundamental logical significance: perhaps there are just objects, and no purpose is served by attempting any finer classification.

To reach such a conclusion is, however, to overlook the crucial question of the kind of sense possessed by proper names or other singular terms for different types of object. We sketched a rough model for the sense of a proper name for a concrete object: to grasp the sense of such a name is to have a criterion of identification of an object as the referent of the name. In many cases, the phrase 'criterion of identification' is too ponderous, and it would be preferable to substitute something like 'propensity for recognition': but the general idea was that a grasp of the sense of the name consisted in a capacity to say, of any given object, whether or not it was the referent or bearer of the name. Here, again, qualification is needed: particularly when the name is a complex one, it may be impossible to determine from mere inspection whether some object presented is that for which the name stands; I understand the term 'Lucy's goldfish' by knowing what is needed, for any given object, to establish that the object is the referent of the term, but, of course, I cannot be expected to be able to determine that question at a glance. At least from Frege's standpoint, there cannot even be a requirement of effective decidability: as long as I can recognize something as settling the question, it is unnecessary that I should be able in all cases to employ some procedure which will lead to a settlement of it.

In offering this account, we were faced with the difficulty of explaining the notion of an object's being 'given' or 'presented': in order to be able to tell, of some object, that it is, or that it is not, the referent of a name, I must have some way of picking out the object that I am identifying otherwise than by the use of the name. In the case of concrete objects, we supposed that this would be done by the use of a demonstrative. On this account, therefore, the understanding of the sense of a name amounts to an ability to determine the truth-value—more properly, to know what would determine the truth-value—of a sentence, containing the name in question, of a quite particular kind, viz. one of the kind we called a 'recognition statement': a sentence of the form 'This is X', where 'X' is the name in question and the 'is' occurs as the sign of identity. Such an ability of course presupposes a grasp of the associated criterion of identity, so that the sense of the name is complex. To grasp its sense, a speaker of the language must, as it were, first know of what kind of object it is the name, i.e. must grasp the criterion of identity associated with it, and then further must be able to know how to recognize a particular object of that kind as the referent of the name.

An account along these lines needs to be modified in any case of a name whose referent is not a possible object of ostension, an object which cannot be presented by the use of a demonstrative accompanied by a pointing gesture (against the background of some understood criterion of identity). Whether or not the distinction between concrete and abstract objects be an apt one, we have already noted that certain objects, of a kind most naturally called 'abstract', cannot be considered as possible objects of ostension. For names of such objects, the account of what it is to grasp the sense of such a name must be revised so as to consider the ability to recognize an object as the referent of the name as relative to some other standard method of being given or presented with such an object. In the case of those abstract objects which can be characterized as being within the range of some functional expression of the kind we have considered, such a method lies to hand: an object of the category determined by any one such functional expression is to be thought of as being given by specifying a particular argument for the function. Where the functional expression is of first level, this will mean specifying an object as argument: e.g. specifying a line of which the object we are concerned with is the direction, or, e.g., a material body of which the object we are concerned with is the shape. It may be that the object which is to be taken as the argument of the function can itself be presented ostensively; if not, then the primitive method of specification of such an object will depend upon the kind of sense possessed by a name for an object of that kind. Where the functional expression is of higher level, e.g. 'the number of Φ's', the argument will not be an object, but a concept, relation or function, and will have to be specified by means of some linguistic expression of the appropriate logical type. We have seen, however, that not every abstract object—a fortiori, not every object which is not a possible object of ostension—can be regarded as within the range of a functional expression of this kind: and so we must seek some more general account of the senses of names.

The notion of what can be an object of ostension is, in practice, nothing like so rigid as we have made it out, in principle, to be. Strictly speaking, a shape cannot be the object of an ostension, in the way that a colour can; and this distinction does mark an important difference between the notion of shape and that of colour, a difference we have taken as a ground for regarding shapes but not colours as abstract objects. But the distinction is very much a difference in principle: we are not ordinarily conscious of the necessity for specifying, of a shape which we are indicating, what we are taking it as the shape of, or of the difference in this regard of a colour that we wish to indicate; not only because we often do specify the object which the colour is the colour of, instead of employing raw ostension, but also because the specification of the object, in the case of shape, is frequently tacit. A letter of an alphabet is, after all, merely a shape—a shape with a particular significance (when what is meant by 'a letter' is a type rather than a token, that is, where 'letter' is being used as when we say that there are twenty-six letters in the English alphabet). But it would be perfectly normal to point to a particular place in a piece of writing or printing, in some script one was not fully familiar with, and ask, 'What is that letter?': the context makes it superfluous to specify what it is that one is taking as forming the letter. When taken as a type, not a token, a move in chess, such as Castling King's side or 1. P-K4 . . . , is surely an abstract object: yet we should feel little oddity about the use of a demonstrative, as in 'What is that move called?' or 'Is that move to be recommended?', even where it was quite clear that 'move' was being used in a type and not a token sense. Conversely, many material objects are not in practice possible objects of ostension, because they are too large and too near, or too small or remote, or simply because they do not affect our senses. One cannot point to the Solar System, though perhaps if space travel progresses sufficiently it may become possible to do so: only an astronaut can point to the Earth. Likewise, one cannot point to a colourless gas; and to smells, sounds and events the notion of pointing hardly seems to apply.

The pointing gesture is an adjunct to the use of a demonstrative when the object indicated emits, reflects or refracts light, and is large enough and near enough to be seen, but far enough and small enough to have a determinate direction. There are several other manners of employing a demonstrative which do not require an ancillary gesture, but can still be regarded as modes of ostension: in one of these, the word 'this' can be construed as meaning something like 'the one we are in' or 'the nearest one'. The occasions on which it is possible to point to a city are comparatively rare: but the phrase 'this city', used to refer to the city in which the speaker is, is extremely common. If one may say, 'this city', or, 'this country', and if such a phrase, though involving no pointing, may count as effecting a species of ostension, then with equal justice the phrases 'this planetary system' and 'this galaxy' may also so count. This form of ostension provides another way in which we may, in the context of our account of the senses of proper names, regard certain objects as being given—given as that object of an understood kind in which the speaker is located or which alone is near at hand to him. And something similar may be held as applying to sounds and smells. If someone asks, 'What is that smell?', he will not use a pointing gesture, since the sense of smell is not directional; and if he asks, 'What is that sound?', he is unlikely to do so, the sense of hearing being so weakly directional: but these too may be taken as types of ostension appropriate to the objects in question, 'that' here being interpretable as meaning something like 'the one currently impinging on our senses'.

It is thus a sufficient condition for something to be a concrete object that it should affect our senses, and can, therefore, be referred to in terms of its sensory impact. This is not, however, a necessary condition, as in the case of a colourless and odourless gas. We might, accordingly, propose as a necessary and sufficient condition that the object be perceptible to some conceivable sensory faculty, though not necessarily one that was even an extension of any of the senses we possess. It would be equally good to say that the presence of the object could be detected by some instrument or apparatus; and this amounts to no more than saying that the object is one which can be the cause of change. More generally, a concrete object can take part in causal interactions: an abstract object can neither be the cause nor the subject of change.

To say that an abstract object cannot be the cause of change seems plausible enough, but the thesis that it cannot be the subject of change is problematic. Cannot the shape of an object change? Cannot the number of sheep on a hill increase? Cannot the centre of mass of Jupiter change position? In the case of number, the answer given by Frege is that there is no one number which is, at all times, the number of sheep on the hill but which is now greater, now smaller, and now vanishes: rather, the change consists in the fact that the number which is the number of sheep on the hill at a certain time is smaller than the number which is the number of sheep on the hill at a later time. (Frege's actual example is the number of inhabitants of Berlin.) We could say the same in the other cases: the centre of mass of Jupiter at one time is not the same point as Jupiter's centre of mass at another time. It is true that we are not absolutely obliged to construe such phrases in Frege's way: we could take, e.g., 'the size of the population of London' as the name of a variable number; but this would have the inconvenience that variable numbers, of this kind, cannot be identified with the numbers that belong to mathematical number-systems, and are quite superfluous, since Frege's method of eliminating them is always to hand. It should be noted, however, that whether or not we speak of objects of a given kind as possible subjects of change is to a marked extent dependent on arbitrary features of the structure of our sentences, or of the analysis which we give of them. If shapes cannot change, and what is called a change of shape is a change, on the part of some object, from having one shape to having another, then the same ought to be said about colours, and we lose our former reason for distinguishing colours as concrete objects from shapes as abstract ones. That reason was that colours can be objects of ostension: but, for a colour to be viewed as a possible object of ostension, it must be allowed to have a position, and hence to be the subject of change, for instance when a colour goes from a given position.

Although Frege's argument may rebut the suggested instances of changes of which abstract objects can be the subject, it still does not show why they cannot be the subjects of other changes. Granted that, when the thirty-first sheep strays on to the hill, there is no number that has increased, is it still not the case that the number 30 has suffered a different change, namely from being the number of sheep on the hill to no longer being the number of sheep on the hill? Inspired by the success of the previous rebuttal, we might attempt to argue that no change has occurred: 30 was, and still is, the number of sheep that were on the hill at a certain time, and is not, and never was, the number of sheep that are on the hill at a later time. But this will not work, since, by this means, we could establish that no change ever takes place: when Robinson grows a beard, he still is a man who was clean-shaven at an earlier date, and he always was a man who would have a beard at that date. If change is to be analysed as Russell suggested, namely as the possession of different truth-values by statements which differ internally only as to their time-references, then presumably a change in an object would be analysed similarly, as a difference in truth-value between two statements containing a term referring to the object, and differing internally only as to their time-references; and, in that case, the above example does illustrate the kind of change which a number may undergo. Our intuitive criterion for what constitutes a change *in* an object is much stricter than this; whether it can be made precise, I do not know, but it is certainly hard to characterize. We might describe an internal change as being one such that Russell's two statements contained no reference to or quantification over any other objects. This would be far too strict: there are certainly relational changes which are nevertheless changes in the object. When a man gets married, or becomes a father, these are certainly changes of which he is the subject; and the paradigmatic example is a change of spatial position. It would still be too strict to admit only those relational changes which actually accompanied an internal change, and too weak to demand only that the relational change be liable to result in an internal one. The closest I can get is that it must be a change which involves some causal interaction between the object and something else; and, in the case of abstract objects, that brings us back to the principle that they cannot take part in causal interactions, which we now see as a ground for speaking in such a way as not to recognize them as possible subjects of change.

Why cannot an abstract object be involved in causal interaction with other objects? It is tempting to go round in a circle, and say that causal interaction always entails some internal change in the objects involved: but this is not always our picture of causal interaction, for instance not in the case of gravitational attraction. It is, rather, a matter of the non-explanatory character of any statement that can be made about the abstract object in itself. If, for the sake of argument, we accept the popular belief that a red rag infuriates a bull, are we to say that the colour causes the bull to charge? There seems no especial reason why we should not: but this is because we are not regarding a colour as an abstract object, and therefore allowing it to have a spatial position, contingently. We can explain the bull's rage by the fact that the colour was there, where he could see it. Contrast the theory that the taste of a substance is determined by the shape of the molecules. Could we say that a certain shape causes a bitter taste? In so far as we regard a shape as a genuine abstract object, and therefore as not having, in itself, a spatial position, but merely as enjoying the property of being the shape of this or that object or configuration, we are reluctant to say this: the taste resulted, not from the presence of the shape, but from the presence of a molecule of that shape. This reflects the fact that shapes are *of* objects in a way that colours are not: it is not merely that we do not choose to say that a shape is in a particular place, whereas we more readily say this of a colour, but that it would make no sense to say that a shape was in a place, without giving some indication of what it was the shape of. A point has, indeed, spatial location, but, in the sense in which a point cannot move, not contingently so. To give a cause of some occurrence, we must cite some contingent fact. (This truistic principle can be accepted without any deep analysis of 'contingent', since all we need is its intuitive sense of 'something that might have been otherwise'; whether or not a precise explanation is possible for this 'might', it is platitudinous that statements of causality yield counterfactual conditionals.) No contingent fact about an abstract object can be cited that cannot more naturally be construed as a fact about concrete ones, for instance, the concrete object which the abstract one is 'of': and hence we do not regard abstract objects as being themselves causally efficacious or the subjects of causal effects.

What is important about abstract objects is not so much the exact line of demarcation between them and concrete ones: it will have become apparent that this line is not clearly marked, and the way we trace it will depend in detail both on the fine structure of our language, in respects easily susceptible to adjustment, and on how we choose to formulate the criterion of distinction. There is, in fact, no reason for wanting a sharp distinction between concrete and abstract objects: the kinds of singular terms which we employ in our language are too variegated for there to be any point in that. The distinction is nevertheless of importance because of the different ways in which the notion of reference applies to names of different kinds. The sense of a proper name is the means by which we recognize an object as its referent. In the case of a name for a concrete object—the case which forms the prototype for the notion of reference—we may equate a grasp of its sense with a capacity to recognize when a recognition statement involving the name has been established, that is, when an object picked out by means of ostension can be identified as the bearer of the name. To the extent that the notion of ostension may be broadened to include reference to an object by its effect on a sense other than sight, its spatial proximity, or some observable causal effect, this model for the sense of a proper name may be extended to names of concrete objects which cannot be seen or are too large or too small to be literally pointed to. An abstract object, on the other hand, can be referred to only by means of a verbal phrase, unaided by any ancillary device for indicating some feature of the environment, save in so far as it is referred to as the value of a function for some concrete object, picked out by ostension, as argument, i.e. by a phrase like 'the shape of *this*'.

It is precisely the fact that we cannot be *shown* an abstract object that prompts the feeling that such objects are spurious, the feeling which underlies nominalism of Goodman's sort. A situation in which an object can be identified as the bearer of a name is one in which we are in some manner confronted by the object. In many cases, the object which is the bearer of a given name may be too remote for such confrontation to take place, or it may have long ceased to exist: but we know what such confrontation would be, and hence feel we have a grasp on the notion that, by means of the name, we make reference to the object. But no confrontation with an abstract object is possible: it is usually not located in space and time, and is not perceptible to sense, not even to imaginable senses transcending ours; and so it is easy to fall into a frame of mind in which we feel that we cannot understand what such objects are supposed to be, and must construct an ontology which excludes them.

In *Grundlagen* Frege expresses the deepest hostility to such tendencies. The philosophical error that he wishes to guard against is not nominalism—the rejection of abstract objects—but psychologism, that is, the interpretation of terms for abstract objects as standing for mental images or other results of mental operations. Such psychologism, he says, springs from the mistake of 'asking after the meaning of a name in isolation from the context of a sentence in which it occurs'; and presumably he would make the same diagnosis of nominalism. If we ask after the meaning of a name of an abstract object in isolation, we are bound to fall back on some mental image, in default of anything else which we could be shown as being the bearer of the name: the corresponding modern mistake would be to conclude that the name did not have any reference at all. On the contrary, Frege says, a name, or any other word, has meaning only in the context of a sentence, and it is only in that context that we may ask after its meaning.

The word here translated by 'meaning' is 'Bedeutung', but Frege had not yet formulated his distinction between reference (Bedeutung) and sense (Sinn), and he never repeated this dictum after the distinction had been formulated. It is therefore possible to interpret the dictum as relating to the *senses* of names and other words. Indeed, it is certainly part of the content of the dictum that sentences play a special role in language: that, since it is by means of them alone that anything can be *said*, that is, any linguistic act (of assertion, question, command, etc.) can be performed, the sense of any expression less than a complete sentence must consist only in the contribution it makes to determining the content of a sentence in which it may occur. For this reason, assigning a bearer to a name would be merely an empty ceremony if it did not serve as a preliminary to introducing a means of using that name in sentences whereby something was said about the object assigned as the bearer. It is precisely because, in Frege's later writings, the unique central role of sentences, which is the key insight embodied in the theory of meaning adumbrated in *Grundlagen*, was so unfortunately lost sight of, that Frege never repeated the dictum that a word has meaning only in the context of a sentence.

This interpretation of the dictum, as a thesis about sense, while exhibiting an important part of its content, does not exhaust it. It is plain from the applications which Frege makes of it that he means more by it than that. One of the ways in which he intends it to be understood is as a defence of contextual definitions. It is possible to miss this, not only because of Frege's later great hostility to contextual definitions, but because of what actually happens in the book about the definition of numerical terms. The example to which Frege principally applies the dictum that we must not ask after the meaning of a name in isolation is that of terms like 'the number 1': yet, when he actually comes to give a definition of such terms, after toying with a form of contextual definition and rejecting it, he gives an explicit definition in terms of classes. It is nevertheless clear that he does not reject the suggested contextual definition simply because it is a contextual definition. He in fact considers three objections to it, the first of which is, in effect, an objection to contextual definitions as such: he rebuts the first two objections, and sustains only the third. It is possible to suppose that subsequent reflection on this third objection led him to think it could be generalized to cover all contextual definitions, and so led him to his later opposition to them: but, in *Grundlagen* itself, it is not offered as a general objection to contextual definitions, but merely as a defect in this particular one. On the contrary, contextual definition is expressly defended in *Grundlagen*, the example being given, with approval, of the definition, in terms of limits, of the standard notation for differentiation.

If the dictum that a word has meaning only in the context of a sentence were intended merely as a thesis about sense, it would tell us nothing about the kind of sense a name can have: it would not follow that there was anything wrong in asking after the sense, or the reference, of a name in isolation, provided that we were aware that the only point of assigning it one was as a preparation for its use in sentences. The use of the dictum as a justification for contextual definition shows, however, that Frege intended more than this: for a name introduced by contextual definition, there simply is no answer to the question what its reference is on its own; all we have is a method of explaining the truth-conditions of any sentence in which it occurs, and Frege is saying that that is all we have a right to demand. We must thus interpret the dictum as expressing, in addition, a thesis about reference: namely that it is illegitimate to suppose that we may always ask to be *shown* the object which is the bearer of a name. Whether or not an expression is a name depends not upon any very precise knowledge of its sense, but merely on its logical role in sentences, namely on those criteria which we attempted to make explicit when originally discussing Frege's notion of a proper name: these criteria may be called 'formal' in a loose sense, relating as they do partly to syntactical questions about the kind of context in which the expression can meaningfully occur, and also to the validity of certain patterns of inference describable in terms of fairly simple transformations of sentences containing the expression. By such criteria, numerical terms, for instance, are readily recognized as proper names, since the structural analogy between, say, '5 is prime' and 'London is noisy', or between '19 is greater than 3' and 'Chicago is west of New York', is easily established. If, then, we have succeeded in determining precise truth-conditions for all possible sentences containing the names in question, we have done everything that is needed in order to give these names a sense. Any further question about whether any such name has a reference or not can be, at most, a question about the truth of an existential statement: just as the question whether the name 'Vulcan' has a reference is an astronomical question, namely as to whether there is a planet whose orbit lies inside Mercury's, so the question whether, say, 'ω_1' has a reference is a mathematical question, namely as to whether there is a least non-denumerable ordinal. The truth of the relevant existential statement is to be determined by the methods proper to that realm of discourse, i.e. in accordance with the truth-conditions that we are supposing have been stipulated for sentences of that kind. There is no further, philosophical, question about whether there *really* exists an object to be the referent of the name.

The thesis expressed by the dictum that a name has meaning only in the context of a sentence, thus interpreted, involves repudiating the conception of a special philosophical sense of 'existence' which would permit us to assert that, in this special sense, numbers do not really exist, while continuing to affirm existential statements of arithmetic, such as that there is a perfect number between 7 and 30. The only sense we have for 'exists' is that given by the existential quantifier in the sentences we ordinarily use: if we have provided determinate truth-conditions for a certain existential statement, and, under those truth-conditions, the statement proves to be true, then there exists something satisfying the condition given in the statement, and that is an end of the matter. This, of course, would not disturb the nominalist, who is not one of those philosophers who wish to eat their cake and have it: he would sternly reject the use of quantifiers taking abstract objects, such as numbers, as their domain, and hence would be unprepared to assent to arithmetical statements at all. But it is also part of Frege's thesis that the nominalist is the victim of a superstition about what has to be done in order to confer a reference on a name. If an expression satisfies the 'formal' criteria for being a name; if it forms part of a vocabulary in which we can construct sentences containing that name and other ones involving quantification to which it is related in the standard way in which a term is related to a quantifier; if we have supplied truth-conditions for these sentences; and if, finally, that one of those sentences which states the condition for the name to have a reference is true according to the truth-conditions we have specified: then there is no further condition which needs to be satisfied for the expression to be a name having a reference.

The superstition which leads to asking after the meaning of a name in isolation, and thus, when it is the name of an abstract object, to either a mentalistic interpretation of it or a nominalist rejection of it as meaningless, is the belief that the sense of a name must be given, or at least always can in principle be given, by a confrontation with the object to which the name is to be attached. The sense of a name of a concrete object may indeed be taken to consist in a criterion for identifying an object as the bearer of the name: but what the nominalist and the mentalist overlook is that such an identification itself depends upon the mastery of some linguistic means, other than the use of the name, for referring to the object, namely the appropriate use of a demonstrative expression. Only language can pick out the object from the total surrounding environment, and can delineate it as an object by imposing a criterion of identity. In the case of the name of an abstract object, this means of referring to the object is lacking: all that we need to master is the use of statements of identity, in which the name occurs on the one side and some other complex term for an object of that kind on the other. The idea that the lack of any means of ostension is fatal to the status of the expression as a proper name, and therefore to the status of its referent as an object, is due to a false picture of the concrete object as something which can, as it were, be given to us on its own, independently of any use of language. Once we grasp that this is not so, then we are no longer disposed to exaggerate the role of ostension. We recognize that, even among concrete objects, there are many different kinds, corresponding to the very different criteria of identity we may employ, and that the notion of ostension is not a univocal one, but must be subjected to various modifications according to the kind of object in question. Having recognized this, we can now see names of abstract objects, which cannot be objects of ostension in however extended a sense, not as a radical departure but merely as a further, and entirely natural, extension of a mode of expression which already exhibits great heterogeneity.

These are undoubtedly the lines along which Frege intends us to understand his notions of proper name and of object, the distinction between concrete and abstract objects being acknowledged but made little of, there being no essential distinction between the uses of names of either kind of object, and, in particular, nothing questionable about the use of names of abstract objects. And, certainly, it can only be by thus stressing the role of names within the context of sentences that it is possible to defend or explain names for abstract objects. The defence nevertheless leaves a residual uneasiness. If the sense of a name is not to be given, or not always to be given, in the form of a criterion for identifying an object as the bearer of the name, how, then, is it to be given? It is one thing to assume airily that we can specify truth-conditions for sentences containing names of abstract objects, e.g. arithmetical statements: but how are these truth-conditions to be specified, if we do not begin by laying down what the reference of the constituent terms is to be? Above all, what becomes of the realism embodied in the use of the name/bearer relation as the prototype of reference and in the principle that the referents of our words are what we talk about? In what sense are we entitled to suppose that abstract objects are constituents of an external reality, when the possession of reference by their names has been interpreted as a matter wholly internal to the language?

There is, indisputably, a considerable tension between Frege's realism and the doctrine of meaning only in context: the question is whether it is a head-on collision. For incomplete expressions, Frege held that the application to them of the notion of reference could be effected only by analogy: but we saw that, while the application could be defended, the analogy appeared to break down at a crucial point. Names of abstract objects are supposed by Frege to have reference in just the same sense as do names of concrete ones: and here we feel disposed to say that it is actually only an analogy, albeit a closer one than with incomplete expressions, and an analogy which calls in question the realist picture of their meaning. We have seen that, for names of abstract objects of certain kinds, it is possible to preserve the structure of the account of the sense of a name as consisting in a criterion for identifying the bearer. We have to find, for a given category of abstract objects, some preferred range of names for them: e.g., in the case of natural numbers, we might select the numerals from some particular system of notation; or, in the case of abstract objects forming the range of some functional expression, such as 'the shape of ξ', whose arguments are possible objects of ostension, we might choose the use of that functional expression completed by a demonstrative. Thus, on such an account, the sense of an arbitrary numerical term ν would consist in the criterion for deciding the truth-value of any sentence of the form $\ulcorner \nu = \varkappa \urcorner$, where $\varkappa$ is a numeral: the criterion for determining such an identity-statement as true would here play the role which was played, in the case of the name of a concrete object, by the criterion for determining the truth of a recognition statement. It must be admitted that no warrant for any such suggestion is to be found in Frege's own writings, and, as an interpretation of Frege, it can be defended only on the plea that he is generally unspecific about sense, except in so far as it can be displayed by the form of definition adopted, when a definition is possible. But, if we do not adopt this suggestion, it is hard to see how the analogy can be made out between the attribution of reference to names of abstract objects and names of concrete ones. If we take seriously the advocacy of contextual definitions in *Grundlagen*, then truth-conditions for sentences containing names for abstract objects can be specified by giving a rule for transforming those sentences into ones which contain no even apparent reference to or quantification over abstract objects of that kind: we can interpret sentences about directions in terms of sentences only about lines. This may quite reasonably be taken as a sound method of justifying the use of names of directions: but it would naturally be construed, not as a demonstration that such names have reference in the same way as names of concrete objects, but as a way of explaining their use without ascribing a reference to them.

Here the conflict between a realist theory of reference and the 'context' doctrine is at its sharpest: if the 'context' doctrine is taken in the very strong sense in which Frege appears to take it in *Grundlagen*, in those passages in which he is commending the use of contextual definition, then it seems to provide a way of dispensing with reference altogether. This conflict is reflected in Frege's hesitancy about contextual definitions. The particular objection which he sustains to the suggested contextual definition of terms of the form 'the number of *F*'s' (and likewise of terms of the form 'the direction of *a*') is that it fails to supply us with a means of determining the truth-value of a sentence of the form 'The number of *F*'s is *c*' (or 'The direction of *a* is *c*'), where '*c*' is a name not of the form 'the number of *G*'s' ('the direction of *b*'). One natural reply might be that we do not need to admit such sentences to our language at all; but, although, at this stage of his career, Frege was evidently willing that not every name should be capable of standing in the argument-place of every predicate, he is sufficiently convinced that objects must be regarded as forming a single category to believe that every identity-statement connecting two names must be given a sense. This scruple surely arises from Frege's realization that there would be an implausibility about the claim that, by means of a contextual definition, a reference had been bestowed upon a certain type of names of abstract objects, if the only way of identifying an object as the bearer of such a name were by means of another name of the same type.

But how is this to be accomplished? Have we not seen that it is a characteristic of the kind of functional expression by means of which we introduce reference to a given range of abstract objects, an expression such as 'the number of Φ's' or 'the direction of ξ', that the objects in its range may not be capable of being referred to except by means of that expression—that we cannot refer to a number save as a number, or to a direction save as a direction? Surely the most we can hope for is that we find a means of specifying that an object not referred to as a number is not a number, one not referred to as a direction is not a direction, rather than that we should be able to find some name by means of which a number or a direction can be picked out otherwise than as a number or a direction.

Frege's solution was, of course, to give an explicit definition of the functional expression in terms of classes. This achieves the original purpose, to find a way of construing numbers or directions as objects that can be identified otherwise than as numbers or directions, namely as classes, and thus, in a way, of making it more plausible that names of numbers and of directions can be regarded as having a reference. As we have seen, the reduction of all abstract objects to classes makes no essential change in the situation: it means merely that there is only one kind of abstract object instead of many. Classes remain objects which can be referred to only as classes; or, if, in accordance with the assimilation of sentences to names, we subsume classes under the more general notion of value-ranges, as is done in *Grundgesetze*, value-ranges remain objects which can be referred to only as value-ranges. Indeed, as we have noted, just this gives rise to concern on Frege's part in *Grundgesetze*: how we are to determine the truth-value of a sentence of the form 'The class of *F*'s is *c*' (more strictly, 'The value-range of *f* is *c*'), where '*c*' is a name constructed without the abstraction operator?

Frege's hesitancy about contextual definitions may thus be seen as due to an only partly conscious realization of the tension between his 'context' doctrine and his realistic notion of reference. Only if we import into Frege's theory of meaning the idea that the sense of the name of an abstract object consists in a criterion for the truth of identity-statements connecting that name with a name of a preferred kind, such as numerals, can we explain the attribution of reference to names of abstract objects at all. But do we, even then, succeed? Can we really justify the claim that abstract objects are constituents of reality in the same sense as concrete ones?

To some extent this depends upon the kind of criterion we admit for the existence of abstract objects. We saw in Chapter 8 that there are divergent views on this question, as representatives of which we took Strawson and Aristotle. For Strawson, whereas the possession of reference by a definite description of an individual (particular) is always a contingent matter, there are definite descriptions of universals which are guaranteed a reference: for instance, 'the colour intermediate between yellow and red'. By contrast, Aristotle held that the existence of anything in a category other than substance always depended upon the existence of a substance in which that thing was or of which it could be predicated. This means that there is a colour or a shape answering to a given description only if there is some object whose colour or shape answers that description. On Aristotle's criterion of existence, it is a contingent (though well-known) fact that there is a colour intermediate between yellow and red, since this depends upon there being some material or visual object whose colour is intermediate between yellow and red: in just the same way, the existence of a shade which fills the gap in Hume's graded sequence of shades of colour is not guaranteed, but is dependent upon there being something which has that shade. From this point of view, Strawson's guarantee of existence is a guarantee only of *possible* existence: we can say a priori that there could be an object which had a shade which would fill the gap, but not that there actually is such an object, nor, therefore, that there actually is such a shade.

It is consonant with Frege's realism that he always adopts an Aristotelian, not a Strawsonian, criterion for the existence of abstract objects. We are tempted to think that the criterion of existence in mathematics is Strawsonian, or, to put the matter in Aristotelian fashion, that mathematics is concerned with possible existence, not with actual existence: for instance, that it is a sufficient ground for the infinity of the series of natural numbers that we can say that, for any n, there could be $n + 1$ objects. Frege has no sympathy with this conception: for him, mathematics is as much concerned with actual existence (construed in an Aristotelian manner) as any other science: if we are to be able to establish the infinity of the number-series, we must be able to show that, for every n, there is some kind F of objects such that there actually are $n + 1$ F's.

This example reminds us, however, that, for Frege, it is not at all the case that all true existential statements are a posteriori, or even synthetic. Of course, there is no reason why Frege, or we, should endorse Kant's dictum that they are all synthetic: for Kant, this thesis followed immediately from his definition of an analytic statement as one in which the predicate is contained in the subject, together with the doctrine that 'exists' is not a predicate; but Frege defined 'analytic' in a much more extensive sense, and, if we accept the notion of analyticity at all, we must follow Frege rather than Kant. But the price of saying that there are necessarily true existential statements is, surely, an obligation to admit that, among the things that exist or may be said to exist, some are not in the world, are not constituents of reality in the same way as concrete objects: at least, this obligation appears to accrue if we hold that there are analytically true existential statements, whatever may be the case for the various alternative senses of 'necessarily true'. I do not know whether this is a principle which could be established, whether, that is, the obscure notion of something's 'being in the world' could be made sufficiently rigorous that a demonstration could be given of the claim: perhaps we are discussing no more than the intuitive acceptability of a picture. But the picture does seem to require that what may be called a 'constituent of reality' is something which can be encountered; and, if the existence of something is an analytic truth, a recognition of its existence can hardly be held to constitute an encounter.

Those abstract objects whose existence Frege takes to be analytic might be called 'pure abstract objects', in analogy with the notion of a pure set. If we start with some collection of individuals (non-sets), we may consider first the totality of all subsets of that collection; if, now, we form the union of these two collections, the individuals and the sets of individuals, we may again form the totality of all subsets of this union, that is, sets whose members may be either individuals or sets of individuals; by once more taking the union of this totality with the collection of individuals, we form the basis for a reiteration of the operation of taking subsets, and so on. If we now consider the denumerably many collections which we have formed in this way, we may take the union of all of them to form the basis for a further operation of forming subsets, and so proceed into the transfinite as far as we wish. This is exactly the cumulative hierarchy of sets which forms the intuitive model for set theories of the Zermelo-Fraenkel type. In general, what sets occur in the hierarchy will depend upon what individuals we started with. Some sets will, however, occur in the hierarchy whatever individuals we start with: they are, in fact, precisely the sets that are generated if we start with no individuals at all. Among such sets are the empty set $\varnothing$, its unit set $\{\varnothing\}$ the set $\{\varnothing, \{\{\varnothing\}\}\}$ containing the empty set and the unit set of its unit set, the set $\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$, and so on. A set can be represented by a tree, with the set itself at the vertex, and each node representing an object (set or individual) whose members, if any, are represented by the nodes immediately below it. Every path in such a tree will be finite, and hence will terminate at some node representing an object which has no members. Such an object must represent either an individual or the empty set. A set is a pure set, that is, one which occurs in the hierarchy whatever collection of individuals we began with, just in case every terminal node in the associated tree represents the empty set.

What objects we recognize the world as containing depends upon the structure of our language. Our ability to discriminate, within reality, objects of any particular kind results from our having learned to use expressions, names or general terms, with which are associated a criterion of identity which yields segments of reality of just that shape: we can, in principle, conceive of a language containing names and general terms with which significantly different criteria of identity were associated, and the speakers of such a language would view the world as falling apart into discrete objects in a different way from ourselves. Thus, in a certain sense, Frege, with his insistence that proper names have sense, and that this sense comprises a criterion of identity, could endorse the second sentence of the *Tractatus*, 'The world is the totality of facts, not things'. Literally taken, this would be wrong for Frege, since, as we have seen, for him facts belong to the realm of sense and not of reference: rather, we should say that, for Frege, the world does not come to us articulated in any way; it is we who, by the use of our language (or by grasping the thoughts expressed in that language), impose a structure on it.

If we use, as Frege does, an Aristotelian criterion for the existence of abstract objects, then the existence of abstract objects of certain kinds will be contingent: there will be no games which are never played, no languages that are never spoken. Of course, set theory provides us with several mechanisms for achieving the effect of quantifying over possible objects without having to invoke modal notions: if, for instance, we want to consider the totality of possible rhyme schemes without restricting ourselves to those that have actually been exemplified, we may first replace rhyme schemes themselves by their representations by sequences of letters such as 'ababb', and then interpret the notion of sequence set-theoretically so that the existence of a sequence does not depend on the physical occurrence of a corresponding string. It nevertheless remains that, in general, the existence of abstract objects depends upon what concrete objects there are: for instance, sets or sequences of concrete objects. But the existence of some abstract objects does not, for Frege, depend on there being any concrete objects at all. The existence of a number satisfying a certain condition does indeed depend upon there being some concept such that the number of objects falling under that concept satisfies the condition in question. But, since it is not only concrete objects but also abstract ones for which we can ask what is the number of those satisfying some given predicate, it is legitimate, in order to establish the existence of a certain number, to cite a concept under which only abstract objects, perhaps only numbers, fall, and in such a way to guarantee the existence of the number quite independently of what concrete objects there are. It is in just this way that Frege proves, to his own satisfaction, the infinity of the series of natural numbers: for any natural number n, the number of numbers less than or equal to n is one greater than n, and hence the series of natural numbers never terminates; the existence of the number 0, from which the series starts, is of course guaranteed by the citation of a concept under which nothing falls.

Abstract objects of this kind, whose existence may be recognized independently of the existence of any concrete objects, and therefore independently of any observation of the world, may be likened to pure sets. When we apply the conceptual apparatus with which language supplies us to reality, this results in the discernment of a variety of objects, concrete and abstract: but the apparatus is such that certain objects will be recognized however the reality is constituted to which we apply it; these are the pure abstract objects, like the natural numbers, whose existence is analytic. This is incomprehensible if we think of the world as composed of objects, as coming to us already segmented into objects: in that case, how could there be a whole plurality of eternally existing, uncreated objects? But, once we realize that our apprehension of reality as decomposable into discrete objects is the product of our application to an originally unarticulated reality of the conceptual apparatus embodied in our language, it should not be particularly surprising that certain objects should result from this operation no matter what the reality is like to which it is applied.

Perhaps not: yet just for that reason it appears impossible to regard the pure abstract objects as constituents of an external reality. At this point, the realistic conception of reference seems to have broken down irrevocably. Pure abstract objects are no more than the reflections of certain linguistic expressions, expressions which behave, by simple formal criteria, in a manner analogous to proper names of objects, but whose sense cannot be represented as consisting in our capacity to identify objects as their bearers. The procedure of determining the truth of certain special kinds of identity-statements, for instance, in the case of natural numbers, equations on one side of which is a numeral and on the other the term in question, may perhaps be taken as embodying the senses of such abstract terms, here numerical ones. As such, it bears a recognizable analogy to the procedure of identifying a concrete object as the bearer of a name, and hence serves to explain the analogy between the logical role of proper names of concrete objects and abstract terms. But precisely the point at which the analogy fails is in the use of the realist picture: the recognition of the truth of a numerical equation cannot be described as the identification of an object external to us as the referent of a term, precisely because there is no sense in which it requires us to discern numbers as constituents of the external world.

Platonism as a philosophy of mathematics has a number of distinguishable strands: while in practice they are most often found together, in principle it would be possible to be a platonist in one respect but not in others. Platonism carries with it a certain picture of what mathematical statements are about: namely that they relate to, and are rendered true or false by, an objective reality external to us just as do statements about the physical universe; but, in the case of mathematical statements, the reality which we are concerned to describe consists of structures of abstract, changeless objects, while statements about the physical world describe structures of temporal, concrete objects. As Frege said, 'the mathematician can no more create anything at will than the geographer can: he too can only discover what is there, and give it a name'. But we cannot tell simply from the picture what the philosophical doctrine is: we have to enquire how the picture is used.

One application of the platonist picture of a mathematical reality external to us is as a means of expressing the conviction that mathematical statements are determined as objectively either true or false, independently of our means of proving or disproving them, just as are statements about the physical universe, on a realist interpretation of those statements. Clearly such a thesis can only be applied to the statements of a theory for which it is thought that there is (up to isomorphism) only one mathematical structure constituting an intended model of the theory. It would not, for example, be inconsistent with a general platonistic outlook to hold that this is not the case in set theory: that we do not have a specific enough intuitive notion of sets for there to be a unique mathematical structure which we may take our set theory as intended to describe, and hence that there are set-theoretic statements—for instance, the continuum hypothesis—which are neither absolutely true nor absolutely false, but about which the whole truth is that they are true in some models and false in others. But, once it is agreed that some intuitive notion, for example, that of 'natural number', has a determinate extension, and that this constitutes the structure which we are aiming to describe by means of some mathematical theory, in this case number theory, then, on this view, statements of that theory are determinately either true or false irrespective of whether we can or ever shall be able to prove or refute them.

Certain very radical forms of constructivism would deny that any intuitive mathematical notion had a determinate extension unless its extension was straightforwardly finite. Thus, according to the strict finitism (ultra-intuitionism) advocated by Essenin-Volpin, it is wrong to suppose that any two intuitive models of the natural numbers are isomorphic: by the 'natural numbers' is meant such a totality as that consisting of those numbers for which it is in practice possible to write down a numeral in some specified notation, and, according to the notation selected, one such structure may be more extensive than another. A related view is that expressed by Wittgenstein in *Remarks on the Foundations of Mathematics*: the sense of an arithmetical predicate, e.g. 'is prime', is given, not by a method that may 'in principle' be used to decide its application, but by the criterion we accept in practice. Since, for any predicate, there exist numbers too large for the practical application of any criterion we possess at any given time, no arithmetical predicate has, at any time, any determinate sense over the whole range of natural numbers. It is a plain consequence of radical views of this kind that we have no right to assume, of an arbitrary arithmetical statement, that it is determinately either true or false.

For any milder form of constructivism, such as intuitionism, however, the objection to assuming the law of bivalence for arithmetical statements is not of this kind. Although, from an intuitionist point of view, natural numbers are mental constructions, and the totality of natural numbers therefore, like any other infinite totality, a potential totality only, since we cannot at any time have carried out infinitely many constructions, it is nevertheless a fully determinate totality: we have a completely specific procedure for generating natural numbers, and it is therefore determined in advance what is and is not to be recognized as being a natural number. In this sense, the intuitive notion of natural number has, intuitionistically as well as platonistically, a fully determinate extension. Likewise, the sense of an arithmetical predicate is, from an intuitionistic standpoint, given by a method which can in principle be used to decide its application, if such a method exists at all: any other criterion which we employ in practice is accepted as a criterion for the application of that predicate only because it has been recognized as yielding an effective method for determining the result of the original decision procedure, in terms of which the sense of the predicate was given.

The reason for the intuitionistic rejection of the conception of determinate truth-values for arithmetical statements lies, rather, in the adoption of a verificationist account of meaning, at least for the language of mathematics. The question turns, therefore, not on the interpretation of numerical terms nor of primitive (decidable) arithmetical predicates, but on the modes of sentence-formation, in the first place quantification. Since quantification over the totality of natural numbers is not an operation on sentences which preserves the property of decidability, a grasp of the senses of the sentences so formed cannot be taken as consisting in a knowledge of the conditions under which they are true or false, but must, rather, be taken to lie in a capacity for recognizing proofs and disproofs of them, since precisely that is what we learn when we learn to use quantified arithmetical statements.

Now it is evident that the status of natural numbers as objects plays, or at least need play, no role in the opposition between these two views. Someone might be convinced that natural numbers were abstract objects, existing eternally and independently of our knowledge of them, and still hold that quantification over them could not be understood as an operation yielding in each case a statement determinately true or false, but only in terms of our capacity for recognizing such a statement as true or as false. Conversely, someone might hold that natural numbers are mental constructions, the product of human thought-processes, and yet accept the interpretation of quantification in terms of infinite logical sum and product, and thus as yielding sentences true or false irrespective of whether we are able to prove them or not. What is in question here is the correct model of meaning, of what we learn when we learn to use the sentences of our language; and, while one model may more naturally go with a picture of the objects in the domain of quantification as existing externally, and the other with a picture of them as mental entities, neither model of meaning either forces on us or is forced on us by the one picture or the other.

A platonistic interpretation of mathematical statements, in so far as this consists merely in the conception that such statements are given meaning by a specification of their truth-conditions, and that we possess a notion of truth for such statements under which each statement (of a sufficiently specific mathematical theory) is determinately either true or false, thus makes no use of the realistic picture of mathematical terms. As Kreisel has remarked, what is important is not the existence of mathematical objects, but the objectivity of mathematical statements. In so far, therefore, as Frege's mathematical platonism amounts to no more than this, his 'context' doctrine of meaning may be accepted as an explanation and defence of the use of abstract terms, but reference may be ascribed to them only as a façon de parler. Their sense may be thought of as a criterion for recognizing certain identity-statements as true, and the analogy between this and the sense of a genuine proper name allowed as explanatory of the formal resemblance between them and proper names. But their meaning cannot be construed after a realistic model, as determined by a relation of reference between them and external objects; for at no point in the explanation of the truth-conditions of sentences in which they occur is there any need to invoke such objects.

To this it might be replied that we require the notion of an abstract object in order to construe the notion of an abstract term, so that the attempt to explain abstract objects as mere reflections of the use of abstract terms achieves nothing. For example, we are proposing to interpret a statement about a natural number, not in Frege's realistic manner, as involving reference to a certain abstract object, over which the arithmetical predicate is defined, but by means of a direct stipulation of the truth-value of sentences resulting from inserting a numeral in the argument-place of the predicate, so that the abstract object is cancelled out. We seemed to be forced into such an interpretation, since we could find nothing to take as constituting identification of a specific number as the bearer of a numerical term save finding the numeral for that number in some preferred system of notation. But, in order to interpret quantification over the totality of natural numbers, we must consider the totality of all numerals in the given system. Numerals, however, cannot be identified with actual written marks, on pain of making the existence of a natural number a contingent matter, and of ensuring that there are only finitely many of them at any time. They have, instead, to be taken as finite sequences of signs, e.g. of digits: and finite sequences are as much abstract objects as natural numbers themselves.

As a demonstration that it is futile to expect to be able to dispense with abstract objects, this argument can appear very powerful, and rightly so: nominalism is indeed crippling to our powers of expression, and only by the most bizarre contortions can a nominalist take even the smallest fragment of mathematics seriously. But, in the present context, the argument overlooks the distinction between abstract objects in general and pure abstract objects, such as Frege took the natural numbers to be. Finite sequences are certainly abstract objects, when we regard them as guaranteed an existence so long as their terms exist; but they are not pure abstract objects if their terms are not. (Here we must construe the notion of a finite sequence as primitive, or possibly as defined by an extension of the standard set-theoretical definition of an ordered pair; not as defined as a function with an initial segment of the natural numbers as domain. We may consider it as a particular primitive method of introducing a kind of abstract object that, given any *n* objects, not necessarily distinct, in a particular order, we think of them as determining a single object which is the sequence having those objects as terms.) Singular terms of various kinds present a gradation according to the extent to which their use involves a mastery of a fragment of the language. Names of the concrete objects we encounter in everyday life stand at one end of the scale, terms for pure abstract objects at the other. While the use of any name requires the mastery of some linguistic technique, so that a grasp of the sense of a name never consists in the bare association of the name with an object presented to us as a separable constituent of reality in advance of all use of language, we may regard the position on the scale as indicating the relative contribution of linguistic and non-linguistic capacities to our having the conception of objects of the kind for which the name stands. In the case of abstract terms of any kind, the fragment of language which has to be mastered in order to learn their use is relatively large, and so the contribution made by the acquisition of linguistic capacities to forming the conception of the objects which they stand for is correspondingly great. But, when the terms are not terms for pure abstract objects, but for, say, shapes of physical bodies or sequences of concrete objects, the use of these terms is still clearly related to processes of observation of the external world and identification of constituents of it. For that reason, therefore, it is still possible to apply to such terms the notion of reference, construed realistically as a relation to something external; although, indeed, the further we travel along the scale, the more stretched becomes the analogy with the prototypical case. It is only when we reach terms for pure abstract objects, however, that the thread snaps completely, and we are concerned with the use of terms which have no external reference at all.

It may be objected, from the other side, that this criticism is unfair to Frege. From a pure mathematician's point of view, natural numbers, considered as the objects of which number theory treats, need have no relation to the external world: but Frege defines the natural numbers in such a way as to display very explicitly their application to empirical reality (as well as to non-empirical reality), their use, namely, to supply answers to empirical questions of the form 'How many . . . ?'. It is therefore entirely unjust to say that, for Frege, natural numbers are pure abstract objects which have no connection with external reality: for Zermelo or von Neumann, no doubt, but not for Frege.

For Frege (in *Grundgesetze*) a number is a class of classes of the same cardinality. If we admitted, as members of such a class of classes, only classes of concrete objects, then we should get what we might call 'actual numbers'. Actual numbers would be neither pure classes nor pure abstract objects: their status would be similar to that of shapes of physical bodies. Frege's cardinal numbers are neither actual numbers nor pure sets, since (apart from 0) all contain classes comprising both abstract and concrete objects among their members. But the notion of a pure abstract object was not meant to be construed by a quite literal analogy with that of a pure set: rather, an abstract object, specified by a certain description, is pure if its existence, under that description, is independent of what concrete objects there are (though its composition may not be). Since Frege wishes to be able to assert the infinity of the number-series independently of the size of the concrete universe, his natural numbers must be understood as pure abstract objects in this sense: they are guaranteed a reference only if to some abstract terms a reference is guaranteed no matter what concrete objects there are.

Sometimes, however, the platonistic picture is put to another use than this. It is held by some platonists that we possess an intuitive apprehension of certain mathematical structures, which guides our formulation of the axioms of the theories which describe them, but may not be, at any given time, fully embodied in those axioms. Such a conception is, for example, invoked to explain the incompleteness of first-order axiomatizations of number theory, in face of our overwhelming inclination to say that we have in mind a unique structure which we are aiming to describe. It is also invoked to suggest that there may be a sense in which set-theoretic statements such as the continuum hypothesis are absolutely true or false, namely relative to an as yet inchoate notion of the kind of model we intend for set theory, a notion which we may later succeed in embodying in new axioms. (The case of set theory of course differs in that the non-categoricity of the axioms cannot be attributed solely to the use of a first- rather than second-order language.)

This is not the place to discuss this contention: we have already made too great an excursion into the philosophy of mathematics. It is plain that other explanations may be possible for the facts which this conception seeks to explain; it is also plain that some acknowledgement must be made of the role of intuitive notions in mathematics. It should be noted that, on this point, the platonist and the constructivist need not be opposed: they merely give different descriptions of the same thing. For the intuitionist, the notion of a mental construction which is the fundamental idea of all mathematics is not one which can be identified with, or even fully represented by, external operations with symbols; the notion of an intuitive proof cannot be expected to coincide with that of a proof in a formal system, and Gödel's incompleteness theorem is thus unsurprising from an intuitionist point of view. What is of importance for us is that platonism of this kind seeks to construe mathematical intuition as playing, with respect to abstract structures, a role analogous to that of perception with respect to physical objects. If such a thesis can be sustained, then indeed the analogy between abstract objects and concrete ones becomes a great deal closer. It ceases to be true that abstract objects are not observable and cannot be involved in causal interaction, since such intuitive apprehension of them may be regarded as just such an interaction. If this analogy can be made out, then all that we have said about pure abstract objects falls to the ground: such objects are possible objects of encounter, and there may be such a thing as identification of such an object which could rightly be compared with the identification of a concrete object. But it is precisely on this possibility of finding some analogue of observation for abstract objects that the tenability of Frege's use of the name/bearer model for abstract terms depends.

Bibliography

The following abbreviations are used for names of journals:

AM—*Annals of Mathematics*
An.—*Analysis*
BPdI—*Beiträge zur Philosophie des deutschen Idealismus*
DL—*Deutsche Literaturzeitung*
JDMV—*Jahresbericht der Deutschen Mathematiker-Vereinigung*
JP—*The Journal of Philosophy*
JSL—*The Journal of Symbolic Logic*
JZN—*Jenaische Zeitschrift für Naturwissenschaft*
PR—*The Philosophical Review*
QJPAM—*The Quarterly Journal of Pure and Applied Mathematics*
RM—*The Review of Metaphysics*
ZPpK—*Zeitschrift für Philosophie und philosophische Kritik.*

The following abbreviations are used for collections of essays:

EF—E. D. Klemke (ed.): *Essays on Frege*, Urbana, 1968.
FLPV—W. V. O. Quine: *From a Logical Point of View*, Cambridge, Mass., 1953.
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E: Frege bibliographies

D (1) contains a bibliography of works by Frege, including translations. *EF*, C (11) and D (2) all contain bibliographies of works both by and about Frege.

F: Works by other authors

This section of the bibliography is intended primarily to enable any reader to find a work referred to in the text. Since I have cited from classical authors—Aristotle, Aquinas, Berkeley, Hume, Kant, Mill, etc.—only well-known views, it has not seemed worth while to include them in the bibliography. On the other hand, every other work mentioned even only in passing has been listed here. Thus this bibliography is not meant to serve as a guide to what has been written about Frege: such guides will be found listed in section (E). Nor is it meant as a guide to the best that has been written about the topics discussed in the book as arising out of consideration of Frege's views: to attempt such a guide would have involved greatly expanding the bibliography, and might also have led to my listing works which I had not read at the time the various chapters were written. On the other hand, I have included a very few works which are not explicitly mentioned in the text, but contain valuable discussions of questions dealt with in it.

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